The Chittagong University Journal of Arts and Humanities
ISSN 1993-5536
VOL.XXXIV, 2018 (pp. 237 - 248)
Published in October, 2023

Zeno's Paradox: A Mathematical Exposition

Dr. Akikul Haque*

Abstract

About 2,500 years ago, the Greek philosopher Zeno of Elea, was a disciple of Parmenides. He asked a simple question that turned out to be exceedingly difficult to answer. He is well known as the author of several ingenious arguments to prove the impossibility of motions. We can ask: how can an arrow travel from the bow of the archer to its target? If it could really move, then, at any place in its (assumed) path, it would be exactly where it is. It would be occupying a space equal to itself. For this reason, there would be no extra space in which to move. Again, at any moment or point of time, it is where it is. The moment is in divisible. The arrow could not be at one place in one part of the moment and at another place at another part of the moment. It simply would have no space or time in which to move. At every point in its trajectory it would be at rest. So, it cannot possibly move. Hence, Zeno concluded that motion is an illusion. The aim of this paper is to explain firstly the Zeno's arguments concerning space and motion, and secondly we would like to analyze it from the mathematical point of view.

1. Zeno's Argument Concerning Space and Motion:

Zeno's paradoxes are primarily about the possibility of motion. Parmenides had combated pluralism. He had declared change and motion to be illusion. Zeno, a strong supporter of the theory of Parmenides, attempted to prove it. According to him, change and motion are impossible even on the pluralistic

* Professor, Department of Philosophy, University of Chittagong

hypothesis. Parmenides denied the existence of the empty space. Zeno tries to support this denial by reducing the opposite view of absurdity. Assume for a moment that there is a space in which things are. If it is something, it will itself be in space. That space will itself be in space, and so on indefinitely. Here we can say that this is absurdity. Things, according to Zeno, are not in space or in any empty void. Therefore, both Parmenides and Zeno were quite right to deny the existence of a void.[1]

Zeno also denied the reality of motion. In support of this statement concerning motion he gave the following argument: **first**, according to him, let us assume that we want to cross a stadium or race course, In order to do so, we would have to traverse an infinite number of points. Moreover, we would have to travel the distance in finite time, if we wanted to get the other side at all. Now the questions arise that how can we traverse an infinite number of points, and so an infinite distance, in a finite time? It can be pointed out that we cannot cross the stadium. In fact, we must conclude that no object can traverse any distance whatsoever, and that all motion is consequently impossible.[2]

Second, let us assume that Achilles and a tortoise are going to have a race, we know that Achilles is a sportsman, so he gives the tortoise a start. Now, by the time that Achilles has reached the place from which the tortoise started, the latter has again advanced to another point. It is important to note that when Achilles reaches that point, then the tortoise will have advanced still another distance, even if very short. We can represent this argument of Zeno by a diagram below:

$\mathbf{T_1 \ldots\ldots.. T_2 \ldots\ldots.. T_3}$

$\mathbf{A_1 \ldots\ldots\ldots.. A_2 \ldots\ldots.. A_3}$

Achilles is always coming nearer to the tortoise. But Achilles never actually overtakes it. T_1 is the starting point of Tortoise and A_1 is the starting point of Achilles. Whenever Tortoise move from T_1 to T_2 then Achilles moves from A_1 to A_2 and so on. This moving process will be going on infinitely (time). From the diagram it is proven that the Tortoise will win the race competition.

The Achilles never can do so, on the supposition that a line is made up of an infinite number of points [i.e, ←**……………..**→], for then Achilles would have to traverse an infinite distance. According to Pythagorean hypothesis[3], Achilles will never catch up the tortoise. Although they assert the reality of motion, the Pythagorean make it impossible on their own doctrine. For it follows that the slower moves as fast as the faster.

Third, let us assume a moving arrow. According to Pythagorean theory the arrow should occupy a given position in space. But to occupy a given position in space is to be at rest. Therefore, the flying arrow is at rest, which is a contradiction.[4] We know the idea that the arrow flies through space. But in order to reach its destination, it must pass over a series of points in space. Hence it must successively occupy these different points [i.e., we assume these points are: X_1, X_2, X_3,..]. Now it is important to note that, to occupy a point of space at given moment, means to be at rest. Here we can say that the arrow is at rest and its moment is but illusory.

The moving arrow goes from distance A to B (figure-1). Here, A is the starting point and B is the last point where it moves. Here AB line is = $X_1+X_2+X_3.$ But the arrow is stationary to its own space, i.e., if we assume that the length of arrow is the same length of $X_1,$ $X_2,$ and X_3 at each points respectively, is stationary to its space.$X_1,$ X_2 and $X_3.$ So it does not move. Therefore, $X_1 = 0$ (Length), similarly $X_2 = 0$ and $X_3= 0$; at every point or its trajectory it would be at rest. Symbolically this can be shown as:

$X_1 = 0$ [arrow is stationary and no move]

$X_2 = 0$ [arrow is stationary and no move]

$X_3 = 0$ [arrow is stationary and no move]

Thus, $AB = X_1+X_2+X_3 = 0 + 0 + 0$

$\therefore AB = 0$ [No space moves]

$\therefore AB = 0$ ….. (I)

But practically (in real distance) $AB \neq 0$ …. (II).

Then, equation (I) and (II) are clearly contradictory. We know that a paradox is generally a puzzling conclusion we seem to be driven towards by our reasoning, but which is highly counterintuitive, nevertheless.

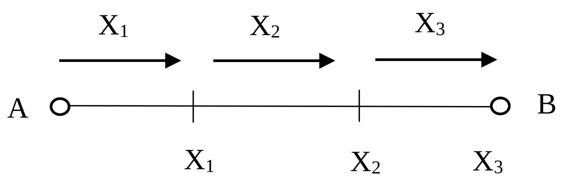

Figure: 1 (moving arrow).

Additionally, if movement takes place, it can take only in space, If space is a reality, it exist somewhere, that is, in a space, which in turn exists in another space, and so on. Thus motion is impossible from every point of view. We cannot suppose it to be real, unless we are willing to affirm an absurdity.

Aristotelean Argument:

Fourth, we know that the fourth argument of Zeno was given by Aristotle. According to this argument, we have to represent to ourselves three sets of bodies on a stadium or race-course. One set is stationary, the other two are moving in opposite directions to one another with equal velocity. Let us consider the below figure:

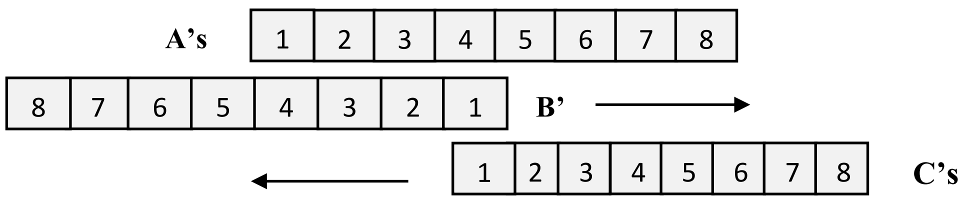

Figure: 2

From the above figure 2, the A's are stationary; the B's and C's are moving in opposite directions with the same velocity. Now they will come to occupy the following positions:

A's	1	2	3	4	5	6	7	8
B's	8	7	6	5	4	3	2	1
C's	1	2	3	4	5	6	7	8

Figure: 3

It is important to realize from the figure 3 that in attaining this second position the front of B_1 has passed four of the A's , while the front of C_1 has passed all the B's. If a unit of length is passed in a unit of time, then the front of B_1 has taken half the time taken by the front of C_1 in order to reach the position of Fig.3. On the other side the front of B_1 has passed all the C's, just as the front of C_1has passed the all the B's. Here the time of their passage must then be equal. We are left then with the absurd conclusion that the half of a certain time is equal to the whole of that time[5].

2. An Explanation from Differential Calculus:

Calculus was invented in the late 1600's by Newton and Leibnitz. Their calculus is a technique for treating continuous motion as being composed of an infinite number of infinitesimal steps. Most mathematicians and physicists believed that continuous motion should be modeled by a function which takes real numbers representing time as its argument and which gives real numbers representing spatial position and its value. This positional function should be continuous or gap-free. In calculus, it is easy to make a distinction between being in motion at a point and being at rest at a point. In order to explain this, first we have to know what the rate of change and derivative is.

Change in dependent variable due to one unit change in independent variable is called the rate of change in the dependent variable with respect to independent variable.

Assume a function $y = f(x)$ in which y depends on x.

Suppose Δx unit change in x causes Δy unit change in y. Therefore, rate of change in y with respect to x is $\frac{\Delta y}{\Delta x}$. Derivative ($\frac{dy}{dx}$) measures the rate of change in y when Δx approaches to zero. In other words, $\frac{dy}{dx} = \text{Limit}_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. Here simply assume the $\frac{dy}{dx}$ and $\frac{\Delta y}{\Delta x}$ are same. This process of finding derivative is called differentiation.[6]

Aristotle had a way of resolving Zeno's paradox. His resolution involved distinguishing between space and time being in themselves divided into parts without limit and simply being divisible without limit. According to him, no magnitude is truly composed of parts. Although it may be divisible into parts without limit, the continuum is given before any such resulting division into parts. Indeed, Aristotle denied that there could be any non-finite parts. This is called a "Finitist". This indicates that non-finite 'parts' cannot be parts of space or time. As he thought that no magnitude can be composed of what has no extension. This means that an arrow can only be "at rest" if it is at the same place at two separate times. For Aristotle, both rest and motion can only be defined over a finite increment of time. Later the notion of an instantaneous velocity came to be accepted. This includes the case where the velocity is zero.[7]

Bishop Berkeley [with Newton] took a skeptical line about the possibility of instantaneous velocities. In the calculation of a derivative, let us consider the following fraction:

$$f(x+\delta x) - f(x) / \delta x,$$

Here δx is a very small quantity. In the elementary case where $f(x)=x^2$, for example, we get

$$(x+\delta x)^2 - x^2/\delta x$$

$$=> x^2 + 2.x.\delta x + \delta x^2 - x^2 / \delta x$$

$$=>2.x.\delta x + \delta x^2 / \delta x \ \dots\dots\dots \ (I)$$

$$=> 2x + \delta x$$

$$\therefore 2x + \delta x \ \dots\dots\dots \ (II)$$

And then to $2x + \delta x$, with δx being subsequently set to zero to get the exact derivative 2x. Berkeley objected that only if δx was not zero could one first divide through by it, and so one was in no position, with the result of that operation, to then take δx to be zero. On the other side, according to Newton's calculus, if it took δx to be zero, it seemed, required the impossible notion of an instantaneous velocity. Aristotle had denied this in connection with his analysis of Zeno's paradoxes.[8]The association between derivative and motion, Newton use the term 'fluxion'. This involved the idea that increment δx was never zero, but merely remained a still finite "infinitesimal".

According to Aristotle, infinites, such as the possible successive division of a line, were only 'potential' not 'actual'. An actual infinite division would end up with non-existensional and non-finite points.

We shall explain Zeno's arrow in the following way by Bertrand Russell. Let us consider the following arrow's figure:

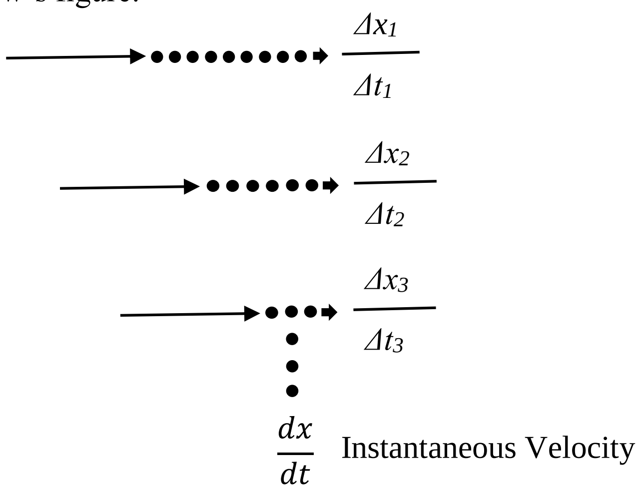

Figure 4: Zeno's Arrow

Let us pick one point in the arrow; we say its center of mass. Then, the motion of the arrow is defined by the position of that point at each moment of time. The calculus defines instantaneous velocity as the derivative of space with respect to time, i.e., $\frac{dx}{dt}$. The value of the derivative at time t is the instantaneous velocity at that time. If $\frac{dx}{dt} = 0$, the arrow is not moving at t.[9]

According to Bertrand Russell, because of the definition of the derivative, this answer to Zeno begs the question. To define the derivative at time $t_0$we consider the distance the object travels in a finite time span Δt, which includes t_0. The ratio $\frac{\Delta x}{\Delta t}$ is its average speed in the in time span Δt. The operation is repeated for smaller and smaller values of Δt. The derivative at the moment t_0 is the limit of the ratio $\frac{\Delta x}{\Delta t}$ as Δt goes to Zero.[10]

For this reason, the definition of the derivative requires consideration of motions over finite stretches of space and time, precisely the kind of motion Zeno claimed to be impossible. To define the derivative, we have to suppose that the conclusion of Zeno's argument is false.[11]This would only evade the problem, not answer it.

Russell then offered an alternative solution of Zeno's paradox. As we have already stated, the motion of the arrow can be represented by nothing the position of its center of mass at each moment in the duration of its flight. Russell proposed an "at-at" theory of motion.[12]

It is important to say that the arrow moving from A to B indicates that it occupies each point in its trajectory at each corresponding moment of time. He does not say that it zips rapidly through these points. If we consider the arrow's state of motion at just one moment, without taking into account its position at any other time, the instantaneous velocity has no meaning.[13]

If we ask how the arrow gets from the beginning of the path A to the midpoint C, Russell answers that it is by occupying each point between these two points at the appropriate time. Again, if we ask how the arrow gets from one point to the next, he reminds us that there is no next point – between any two points in its continuous path there are infinitely many others.[14]

According to Salmon, Russell's solution to the arrow paradox is completely satisfactory. In addition, it indicates an analogous approach to the concept of causal transmission. According to him, instead of an arrow, for example, we think of a bullet shot from a gun, when the bullet leaves the gun, marks are made upon it that enable experts to identify the gun from which the bullet was shot. The moving bullet is a causal process; the marks are transmitted. Once the marks have been imposed by the interaction of the bullet with the gun, they remain on the bullet as it travels. It is important to note that the mark is transmitted indicates that it is at the appropriate place in the process at the appropriate time. Moreover, the bullets transmit mass, a conserved quantity in this non -relativistic context. It possesses a certain mass when it exists from the gun. It continues to possess that same mass without any further interactions to replenish mass. Here the mass in question is at the appropriate place at the appropriate stage in the evolution of this process. For this reason, we can adapt Russell's "at-at" theory of motion to an "at-at" theory of causal transmission.[15]The significance of this theory is that the bullet transmits information. The marks identify the gun from which it was shot. According to Salmon, it also transmits causal influence. If the bullet strikes a person, it will produce a wound – possibly a fatal wound.

3. Mathematical Exposition:

The oldest 'solution' to the paradox was done from a purely mathematical perspective. The claim admits that, there might be an infinite number of jumps that we would need to take. But that each new jump got smaller and smaller than the prior one. Therefore, as long as we could demonstrate that total sum of every jump we need to take adds to a finite value, it doesn't matter how many chunks we divide it into. For example, if the total journey is defined to be 1 unit, then we could get there by adding half after half after half, etc. The series 1/2+1/4+1/8+…does indeed converge to 1, so that we wind up covering the entire needed distance if we add an infinite number of terms.

Suppose, series =1/2+1/4+1/8+…

$$2 * \text{series} = 2[1/2+1/4+1/8+\ldots]$$

$$= 1 + [1/2+1/4+1/8+\ldots]$$

Therefore, [2*(series)-(series)] = (1+1/2+1/4+1/8+…)-(1/2+1/4+1/8+…)

=1.

But it is also flawed. This mathematical line of reasoning is only good enough to show that the total distance we must travel converses to a finite value. It does not tell us anything about how long it takes us to reach our destination; this is the tricky part of the paradox [16].

4. Concluding Remarks:

Many thinkers, both ancient and contemporary, tried to resolve this paradox by invoking the idea of time. According to Ethan Siegel, pure mathematics alone cannot provide a satisfactory solution to the paradox[17]. He argues that the reason is simple: the paradox isn't simply about dividing a finite thing up into an infinite number of parts, rather about the inherently physical concept of a rate. Although the paradox is usually posed in terms of distance alone, the paradox is really about motion, which is about the amount of distance covered in a specific amount of time. He also argues that objects can move from one location to another in a finite amount of time. Their velocity are not always finite, but because they don't change in time unless acted upon by an outside force. This is basically Newton's first law[18], but applied to the special case of constant motion. There is an explicit relationship between distance, velocity and time. So, motion from one place to another is possible. This is still an interesting exercise not only for philosophers and mathematicians, but also physicists who have extended it to quantum phenomena to resolve Zeno's paradox.

The arguments of Zeno are mere sophistries on the part of Zeno. These arguments are ingenious tricks. They err by supposing that a line is composed of points and time of discrete moments. It may be that the solution of the riddles is to be found in showing that the line and time are continuous and not discrete. In this sense, Zeno was not concerned to hold that they are discrete. On the other hand, he is concerned to indicate absurd results which follow from supposing that they are discrete. Zeno believed that motion is an illusion and is impossible. The assumption of its possibility leads to contradictory and absurd conclusions. According to Copleston, this hypothesis of Zeno does nothing to explain motion, but only leads one in absurdities.[19]For this reason, Zeno reduced the hypothesis of his adversaries to absurdity. This indicates that the real result of his dialectic was not so much to establish Parmenidean monism.

Aristotle' resolving Zeno's paradoxes convinced most people until more recent times. His notion of an instantaneous velocity came to be accepted. But Zeno's paradoxes lead us into twentieth century developments in the area of logical paradoxes, a branch of formal logic, intensional logic, and mathematics.

Notes and References:

1. Fredrick Copleston, S. J. *A History of Philosophy*, Vol.1, Search Press, London. 1946. P. 56.

2. *Loc.cit*., also see, (c.f) Aristotle, *Physics*, Z $_9$, 239 b $_9$; 2, 233 a 21; Top. , 8160 b $_7$.

3. The Pythagoreans hypothesis is that reality is made up of units. These units are either with magnitude or without magnitude. Suppose, a line is made up of units possessed of magnitude. The line will be infinitely divisible and the units will still have magnitude and so be divisible. In this case the line will be made up of an infinite number of units, each of which is possessed of magnitude. As a result, the line must be infinitely great, as composed of an infinite number of bodies.

4. Copleston, *Ibid*, P. 57.

5. *Ibid.* Pp. 57-58.

6. Mohammed Saiful Islam, *Microeconomics*, 3rd ed., Sumaiya Publication, Chittagong, 2017, P. 21.

7. 'Classical Logical Paradoxes' *Internet Encyclopedia of Philosophy*. [www.https://iep.ulm.edu/per-log/Logical Paradoxes] dated: 23.03.2022, time 2.00pm.

8. *Ibid.*

9. Salmon. Wesley C, ‘Causation’, in Richard M. Gale (ed.) *The Blackwell Guide to Metaphysics,* Blackwell Publishers Ltd; Oxford, UK, 2002, P.34.

10. *Loc. cit*.

11. *Loc. cit*.

12. *Loc. cit*.

13. Russell Claims that these considerations actually vindicate Zeno. However, Russell points out, even if the arrow is at rest at each point of its trajectory, it does not follow that it is always in the same place.

14, Salmon,W C., *Op. cit.* P. 34.

15. *Loc. cit*.

16. Ethan Siegel, ‘This is How Physics, Not Math, Finally Resolves Zeno’s Famous Paradox’, [https// :forbes.com/sites/starts with bang/2020/05/05/this-is-how-physics-not-math-finally-resolves-Zeno’s-paradox-?sh=4f$_0$11b1233f$_8$, dated:22-06-2022,time 10 am.

17. *Loc. cit.*

18. Newton’s first law: objects at rest remain at rest and objects in motion remain in constant motion unless acted on by an outside force.

19. Copleston, *Ibid*., P. 58.